Martingale theory

M1, TSE

Reminders

Reminders

Definition

A process $(X_n, \mathcal{F}_n)_n$ is a martingale (res. sub-martingale, super-martingale) if

- $\mathcal{F}_n$ is a filtration and $(X_n)_n$ is adapted to $(\mathcal{F}_n)_n$
- $\forall n \in \mathbb{Z}_+, X_n$ is integrable
- $\forall n \in \mathbb{Z}_+ : \mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$ (resp. $\geq X_n$, $\leq X_n$.)

Proposition

1. Let $(X_n, \mathcal{F}_n)$ be a sub-martingale
2. Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex and non-decreasing.

If $\forall n \in \mathbb{Z}_+ : \mathbb{E}\left[\left|\phi(X_n)\right|\right] < \infty$, then $(\phi(X_n), \mathcal{F}_n)_n$ is a sub-martingale.

Reminders

Definition

Let $(\mathcal{F}_n)_n$ be a filtration. A random variable T taking values in $\mathbb{Z}_+ \cup \{\infty\}$ is a stopping time (or optional time) if

$$\forall n \in \mathbb{Z}_+ : \ \{T = n\} \in \mathcal{F}_n.$$

Definition

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_n, \mathbb{P})$ be a filtered probability space. The σ-field generated by a stopping time T is

$$\mathcal{F}_T = \Big\{ \Lambda \in \mathcal{F} \,\Big|\, \forall k \in \mathbb{Z}_+ : \Lambda \cap \{T = k\} \in \mathcal{F}_k \Big\}.$$

Reminders

Proposition

- A random variable T with values in $\mathbb{Z}_+ \cup \{\infty\}$ is a stopping time if and only if $\{T \leq n\} \in \mathcal{F}_n, \forall n \in \mathbb{Z}_+$.
- If $(X_n, \mathcal{F}_n)_n$ is a stochastic process and B is a Borel set, then

$$T = \inf \left\{ n \in \mathbb{Z}_+ : X_n \in B \right\}$$

  is a stopping time.
- If T_1 and T_2 are stopping times, then so are $T_1 \wedge T_2$ and $T_1 \vee T_2$.
- If $T_1 \leq T_2$, then $\mathcal{F}_{T_1} \subseteq \mathcal{F}_{T_2}$.
- If T is a stopping time, then T is $\mathcal{F}_T$-measurable.

Reminders

Theorem

Let $(X_n, \mathcal{F}_n)_{n\in\mathbb{Z}_+}$ be a submartingale, and let T be a stopping time, finite or infinite. Then $(X_{n\wedge T}, \mathcal{F}_n)_{n\in\mathbb{Z}_+}$ is also a submartingale, as is $(X_{n\wedge T}, \mathcal{F}_{n\wedge T})_{n\in\mathbb{Z}_+}$.

Corollary

Let $(X_n, \mathcal{F}_n)_{n\in\mathbb{Z}_+}$ be a martingale, and let T be a stopping time. Then $(X_{n\wedge T}, \mathcal{F}_n)_{n\in\mathbb{Z}_+}$ and $(X_{n\wedge T}, \mathcal{F}_{n\wedge T})_{n\in\mathbb{Z}_+}$ are also martingales.

To show that $(X_{n\wedge T})_n$ is a martingale, is suffices to show that $(X_n)_n$ is a martingale and T is a stopping time!

Doob's optional sampling theorem

Doob's optional sampling theorem

Theorem

Let $X = (X_n)_n$ be a $\mathcal{F}_n$-martingale (resp. super-martingale) and T be a stopping time. If *one* of the following three properties holds

(A1) T is bounded a.s.,

(A2) T is finite and X is bounded a.s.,

(A3) T is integrable ($\mathbb{E}(T) < \infty$) and $\exists c > 0 : \sup_n |X_n - X_{n-1}| \leq c$ a.s.,

then

$$\mathbb{E}(X_T) = \mathbb{E}(X_0) \qquad \text{(resp. } \mathbb{E}(X_T) \leq \mathbb{E}(X_0)\text{)}.$$

Interpretation: Under some conditions, there is no strategy that can transform a fair game into a profitable one (such that $\mathbb{E}(X_T) > \mathbb{E}(X_0)$) using only the information available so far.

Elements for the proof

Reminder: Dominated convergence theorem

Let $(X_n)_n$ be a stochastic process such that

- There exists a random variable X_∞ such that $X_n \to X_\infty$ a.s.
- There exists an integrable random variable Y independent of n (called dominating random variable) such that

$$\forall n \in \mathbb{Z}_+ : |X_n| \leq Y \quad a.s.$$

Then $\lim_{n\to\infty} \mathbb{E}(X_n) = \mathbb{E}[X_\infty]$.

Generalization under Assumption (A1)

The previous theorem under Assumption A1 (T is bounded a.s.) can be generalized as follows.

Theorem

Let $(X_n, \mathcal{F}_n)_{n\in\mathbb{Z}_+}$ be a sub-martingale, and let $S \leqslant T$ be bounded stopping times. Then (X_S, X_T) is a sub-martingale relative to the filtration $(\mathcal{F}_S, \mathcal{F}_T)$.

To understand this result, define

$$Y_0 = X_S, \qquad \mathcal{G}_0 = \mathcal{F}_S$$
$$Y_1 = X_T, \qquad \mathcal{G}_1 = \mathcal{F}_T.$$

The theorem says that (Y_0, Y_1) is a submartingale adapted to $(\mathcal{G}_0, \mathcal{G}_1)$, which implies

$$\mathbb{E}[X_T|\mathcal{F}_S] \geq X_S.$$

Connection with Doob's sampling theorem

Link with Doob's sampling theorem under Assumption A1?

- The previous theorem holds for super-martingales as well, so it holds for martingales.
- Apply this theorem in the martingale case with $S = 0$, to conclude $\mathbb{E}[X_T|\mathcal{F}_0] = X_0$, hence $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

Generalization under Assumption (A2)

Doob's optional sampling theorem under Assumption A2 ($T < \infty$ and X is bounded a.s.) can be generalized as follows.

Corollary

Let $(X_n, \mathcal{F}_n)_{n \in \mathbb{Z}_+}$ be a bounded sub-martingale and let $S \leqslant T$ be finite stopping times. Then (X_S, X_T) is a submartingale relative to the filtration $(\mathcal{F}_S, \mathcal{F}_T)$.

Doob's optional sampling theorem under Assumption A2 is again a special case where $S = 0$.

Elements for the proof

Elements to have in mind for the proof

- $\mathbb{E}[X|\mathcal{G}]$ is the $\mathcal{G}$-measurable random variable Z such that

$$\forall \Lambda \in \mathcal{G} : \quad \int_\Lambda X d\mathbb{P} = \int_\Lambda Z d\mathbb{P}.$$

- If X, Z are $\mathcal{F}$-measurable, then

$$X \leq Z \quad \Longleftrightarrow \quad \forall \Lambda \in \mathcal{F} : \int_\Lambda X d\mathbb{P} \leq \int_\Lambda Z d\mathbb{P}.$$

- Let T be a stopping time, $\Lambda \in \mathcal{F}_T$ and $m \leq n$, then $\Lambda \cap \{T = m\} \in \mathcal{F}_{T \wedge n}$.

Generalization under Assumption (A2)

The previous corollary requires that X be bounded. We can extend it to processes that are bounded on one side only.

Corollary

Let $(X_n, \mathcal{F}_n)_{n\in\mathbb{Z}_+}$ be a ≤ 0 submartingale (resp. ≥ 0 supermartingale) and let $S \leqslant T$ be finite stopping times. Then (X_S, X_T) is a submartingale (resp. supermartingale) relative to the filtration $(\mathcal{F}_S, \mathcal{F}_T)$.

Warning: This does NOT imply that a non-negative martingale remains a martingale under optional sampling.

Counter example?

Counter example

Let $Y_1, Y_2, \ldots \overset{iid}{\sim} \mathrm{Ber}(p)$ and define

$$X_n := \prod_{j=1}^{n} \frac{Y_j}{p}, \qquad X_0 = 1.$$

This defines a martingale with respect to $(\sigma(Y_1, \ldots, Y_n))_{n \in \mathbb{Z}_+}$ (why?).

Let $S = 0$ and $T := \inf\{n : X_n = 0\}$, which are almost surely finite.

Then $X_S = 1$ while $X_T = 0$, so (X_S, X_T) is not a martingale.

Elements for the proof

Reminder: Monotone convergence theorem

Let $(X_n)_n$ be a sequence of non-negative (resp. non-positive) random variables.

Suppose that, $\forall n \in \mathbb{Z}_+ : X_{n+1} \geq X_n$ (resp. $X_{n+1} \leq X_n$). Then it holds that

$$\mathbb{E}\left[\lim_{n\to\infty}\uparrow X_n\right] = \lim_{n\to\infty}\uparrow \mathbb{E}\left[X_n\right]$$

$$\text{(resp.} \quad \mathbb{E}\left[\lim_{n\to\infty}\downarrow X_n\right] = \lim_{n\to\infty}\downarrow \mathbb{E}\left[X_n\right]).$$

Application: Gambler ruin's problem

Gambler's ruin

A player starts with x_0 euros and earns or looses €1 at every step.

Question: let a, b be integers such that $a < x_0 < b$. What is the probability that the gambler reaches a before b?

For any $k \in \mathbb{Z}_+$, let $X_k = x_0 + \sum_{i=1}^{k} Y_i$ where $Y_i \overset{iid}{\sim} \mathrm{Rad}\left(\frac{1}{2}\right)$.

Example: if $a = 0$, the player is ruined. Probability that they are ruined before reaching a target b?

Gambler's ruin

Let $(\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ be the filtration generated by $(X_n)_{n\in\mathbb{Z}_+}$.

Now define the finite stopping time

$$T := \inf\{n \geqslant 0 : X_n = b \text{ or } a\}$$

Now $(X_n)_{n\in\mathbb{Z}_+}$ is a martingale, so the stopped process $(X_{n\wedge T})_{n\in\mathbb{Z}_+}$ is also a martingale, that is bounded by a and b.

We can apply the the stopping theorem to the stopping times $S \equiv 0$ and $T :$ (X_0, X_T) is a martingale.

$$\begin{aligned} x_0 = \mathbb{E}[X_0] = \mathbb{E}[X_T] &= p\,\mathbb{E}[X_T|X_T = a] + (1-p)\mathbb{E}[X_T|X_T = b] \\ &= pa + (1-p)b. \end{aligned}$$

Hence $p = \frac{b-x_0}{b-a}$

How long does it take to hit a or b? Let's compute $\mathbb{E}(T)$.

First, we claim that $(Z_n)_{n\in\mathbb{Z}_+}$, given by $Z_n := X_n^2 - n$ for all n, is a martingale. Indeed,

$$\begin{aligned}\mathbb{E}\left[Z_{n+1} - Z_n \mid \mathcal{F}_n\right] &= \mathbb{E}\left[X_{n+1}^2 - n - 1 - X_n^2 + n \mid \mathcal{F}_n\right] \\ &= \mathbb{E}\left[\left(X_{n+1} - X_n\right)^2 + 2X_n\left(X_{n+1} - X_n\right) - 1 \mid \mathcal{F}_n\right]\end{aligned}$$

Now $\left(X_{n+1} - X_n\right)^2 = 1$, so the above term is equal to

$$2\mathbb{E}\left[X_n\left(X_{n+1} - X_n\right) \mid \mathcal{F}_n\right] = 2X_n\mathbb{E}\left[X_{n+1} - X_n \mid \mathcal{F}_n\right] = 0$$

since X is a martingale. This verifies the claim.

Once again we apply the system theorem to the bounded stopping times $S \equiv 0$ and $T \wedge N$ for a integer N. Thus

$$x_0^2 = \mathbb{E}\left[Z_{T\wedge N}\right] = \mathbb{E}\left[X_{T\wedge N}^2 - T\wedge N\right]$$

so that for each N

$$\mathbb{E}[T\wedge N] = \mathbb{E}\left[X_{T\wedge N}^2\right] - x_0^2 \tag{25}$$

As $N \to \infty$, we have $T \wedge N \to T$ and $X_{T\wedge N} \to X_T$ boundedly, since $a \leqslant X_{T\wedge N} \leqslant b$, so we can take limit on both sides, using monotone convergence on the left and bounded convergence on the right. Thus $\mathbb{E}[T] = \mathbb{E}\left[X_T^2\right] - x_0^2$. We found the distribution of X_T above, so we see this is equal to

$$\frac{x_0 - a}{b - a} b^2 + \frac{b - x_0}{b - a} a^2 - x_0^2 = (b - x_0)(x_0 - a)$$

Thus $\mathbb{E}[T] = (b - x_0)(x_0 - a)$.

What if $b = \infty$? I.e., the gambler keeps playing until they hit 0.

As $b \uparrow \infty$, we have $T \uparrow T_a = \inf\{n : X_n = a\}$.

By monotone convergence, we deduce (recall that $x_0 > a$)

$$\mathbb{E}[T_a] = \lim_{b \to +\infty} (b - x_0)(x_0 - a) = +\infty$$

The gambler will eventually go bankrupt (namely $T_a < +\infty$ a.s., as we will see later on), but it will take infinite expected time!

The maximal inequality

Recall (Chebyshev): $\lambda \mathbb{P}[|X| \geqslant \lambda] \leqslant \mathbb{E}[|X|]$.

Martingales satisfy a similar, but much more powerful inequality.

Theorem

Let $(X_n)_{n \in [\![0,N]\!]}$ be a non-negative submartingale. Then

$$\lambda \mathbb{P}\left[\max_{n \in [\![0,N]\!]} X_n \geqslant \lambda\right] \leqslant \mathbb{E}\left[X_N\right] \tag{26}$$

Remark: If $(X_n)_{n \in [\![0,N]\!]}$ is a martingale, $(|X_n|)_{n \in [\![0,N]\!]}$ is a non-negative submartingale, and we have $\lambda \mathbb{P}\left[\max_{n \in [\![0,N]\!]} |X_n| \geqslant \lambda\right] \leqslant \mathbb{E}\left[|X_N|\right]$, which is the extension of Chebyshev's inequality we mentioned above.

The upcrossing inequality

Let $x_0, x_1, \dots, x_N$ be a finite sequence of real variables. Define the number of upcrossings of an interval $[a, b]$ as the number of times the sequence goes from below a to above b.

Set $\alpha_0 := 0$,

$$\alpha_1 := \begin{cases} \inf\{n \leqslant N : x_n \leqslant a\} & \\ N+1, & \text{if there is no such } n \end{cases}$$

and for $k \geqslant 1$,

$$\beta_k := \begin{cases} \inf\{n \geqslant \alpha_k : x_n \geqslant b\} & \\ N+1, & \text{if there is no such } n \end{cases}$$

$$\alpha_{k+1} := \begin{cases} \inf\{n \geqslant \beta_k : x_n \leqslant a\} & \\ N+1, & \text{if there is no such } n \end{cases}$$

If $\beta_k \leq N$ then $x_{\alpha_k} \leq a \leq b \leq x_{\beta_k}$: upcrossing of $[a, b]$ between α_k and β_k.

Definition (Number of upcrossings)

The number of upcrossings of the interval $[a, b]$ by the sequence $x_0, x_1, \ldots, x_N$ is $\nu_N(a, b) := \sup\{k \in [\![0, N]\!] : \beta_k \leqslant N\}$.

Definition (Number of upcrossings by a process)

Let $(\mathcal{F}_n)_{n \in [\![0,N]\!]}$ be a filtration and $(X_n)_{n \in [\![0,N]\!]}$ be adapted to it.

We define $\nu_N(a, b)$ as the number of upcrossings of $[a, b]$ by $(X_n)_n$.

Theorem (Upcrossing inequality)

Let $(X_n, \mathcal{F}_n)_{n \in [\![0,N]\!]}$ be a submartingale and let $a < b$ be real numbers. Then the number of upcrossings satisfies

$$\mathbb{E}\left[\nu_N(a,b)\right] \leqslant \frac{\mathbb{E}\left[(X_N - a)_+\right]}{b-a}$$

Elements to have in mind for the proof

Let $(X_n)_n$ be $(\mathcal{F}_n)_n$-adapted.

1. (Exercise 2.7) If S is a stopping time and $B \subseteq \mathbb{R}$ is a Borel set, then $T = \inf\{n > S : X_n \in B\}$ is a stopping time.
2. If $(X_n, \mathcal{F}_n)_n$ is a submartingale and $\alpha \leq \beta$ are a bounded stopping times, then $\mathbb{E}[X_\beta - X_\alpha] \geq 0$.

The above result extends to an infinite parameter set. If $X := (X_n)_{n\in\mathbb{Z}_+}$ is a process, define $\nu_\infty(a,b) := \lim_{N\to\infty} \nu_N(a,b)$. Then the upcrossing inequality extends to the following.

Corollary

Let $(X,\mathcal{F}) := (X_n,\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ be a submartingale and let $a < b$ be real numbers. Then the number of upcrossings $\nu_\infty(a,b)$ of $[a,b]$ by X satisfies

$$\mathbb{E}\left[\nu_\infty(a,b)\right] \leqslant \frac{\sup_{N\in\mathbb{Z}_+} \mathbb{E}\left[(X_N - a)_+\right]}{b-a}$$

Reminders for the proof

Elements to have in mind for the proof

(Monotone Convergence Theorem)

If $\forall n \in \mathbb{Z}_+ : X_{n+1} \geq X_n \geq 0$ and X_n is measurable, then

$$\mathbb{E}\left[\lim_{n\to\infty}\uparrow X_n\right] = \lim_{n\to\infty}\uparrow \mathbb{E}\left[X_n\right]$$

Martingale convergence

The results on the upcrossing numbers allow us to obtain this important martingale convergence theorem.

Theorem (Martingale Convergence Theorem)

Let $(X_n, \mathcal{F}_n)_{n \in \mathbb{Z}_+}$ be a submartingale and suppose that

$$\sup_{n \in \mathbb{Z}_+} \mathbb{E}\left[|X_n|\right] < \infty.$$

Then there exists a finite integrable r.v. X_∞ such that

$$\lim_{n \to \infty} X_n = X_\infty \quad \text{a.s.}$$

Thus, to show convergence of a (sub/super-)martingale, it suffices to show that its absolute expectation is bounded.

Reminders for the proof

Elements to have in mind for the proof

1. If $Z \geq 0$ and $\mathbb{E}(Z) < \infty$ then $\mathbb{P}(Z = \infty) = 0$, i.e. $Z < \infty$ a.s.
2. The set $\mathbb{Q}^2$ is countable
3. If $\mathcal{I}$ is countable and $(A_n)_{n \in \mathcal{I}} \in \Omega^{\mathcal{I}}$ is a sequence of measurable events, then

$$\mathbb{P}\Big(\bigcup_{n \in \mathcal{I}} A_n\Big) \leq \sum_{n \in \mathcal{I}} \mathbb{P}(A_n).$$

4. (Fatou's lemma) If $X_n \in [0, \infty]$ for any $n \in \mathbb{Z}_+$, then

$$\mathbb{E}\left[\liminf X_n\right] \leq \liminf \mathbb{E}\left[X_n\right]$$

Corollary

A non-negative supermartingale and a non-positive submartingale converge with probability one.

Application: Gambler ruin random walk $(X_n)_{n\in\mathbb{Z}_+}$. We stated without proof that $(X_n)_{n\in\mathbb{Z}_+}$ eventually reached the complement of any finite interval. Let us show that for any a, there exists $n \in \mathbb{Z}_+$such that $X_n \leqslant a$.

Application: Gambler's ruin

Consider the stopping time $T = \inf\{n : X_n \leqslant a\}$. Then $(X_n)_{n\in\mathbb{Z}_+}$ is a martingale, $X_0 = x_0$, and $|X_{n+1} - X_n| = 1, \forall n \in \mathbb{Z}_+$.

The process $(X_{n\wedge T})_{n\in\mathbb{Z}_+}$ is a martingale and $X_{n\wedge T} \geq a \wedge x_0$, so it converges a.s. by the Martingale Convergence Theorem.

This imposes $T < \infty$, since $\left[n < T \Longrightarrow \left|X_{n+1} - X_n\right| = 1\right]$, so convergence is impossible on the set $\{T = \infty\}$.

But if $T < \infty$ then $X_T \leqslant a$, so the process eventually goes below a.

Gambler's ruin

It is a rather easy consequence of this to see that $(X_n)_{n\in\mathbb{Z}_+}$ must visit all integers, non-negative or negative.

In fact, it must visit each integer infinitely often, and with probability one, both $\liminf_{n\to\infty} X_n = -\infty$ and $\limsup_{n\to\infty} X_n = +\infty$.

Convergence in L^2

Boundedness in L^2

Definition

A martingale $M = (M_n)_n$ is bounded in L^2 if $\sup_n \mathbb{E}\left[M_n^2\right] < +\infty$.

We start by showing the following proposition

Proposition

Assume $(M_n)_n$ is bounded in L^2 and let $\Delta M_k = M_k - M_{k-1}$ for any $k \in \{1, \ldots, n\}$. For any $n \in \mathbb{Z}_+$, it holds

$$\mathbb{E}(M_n^2) = \mathbb{E}[M_0^2] + \sum_{k=1}^{n} \mathbb{E}[(\Delta M_k)^2]$$

Thus $(\mathbb{E}(M_n^2))_n$ is non-decreasing and bounded, hence it converges.

Boundedness in L^2

For $k \geq 1$, we define the increments by $\Delta M_k = M_k - M_{k-1}$.

For $1 \leq i < j$, we have

$$\mathbb{E}\left[\Delta M_i \Delta M_j\right] = 0$$

This implies

$$\mathbb{E}\left[M_n^2\right] = \mathbb{E}\left[M_0^2\right] + \sum_{k=1}^{n} \left(\Delta M_k\right)^2$$

which proves the convergence of the sequence $\left(\mathbb{E}\left[M_n^2\right]\right)_n$ to a finite positive value. This observation will be crucial to prove the following theorem.

Convergence in L^2

The following theorem states that a martingale bounded in L^2 converges in L^2.

Theorem

Let $M = (M_n)_n$ be a bounded martingale in L^2. Then, there exists a square-integrable random variable $M_\infty \in L^2$ such that

$$\lim_{n \to \infty} \mathbb{E}\left[(M_n - M_\infty)^2\right] = 0.$$

Elements to have in mind for the proof

- A sequence $(X_n)_n$ is a Cauchy sequence in L^2 if

$$\forall \varepsilon > 0,\ \exists n \in \mathbb{N},\ \forall p \geq n:\ \mathbb{E}\left[(X_p - X_n)^2\right] \leq \varepsilon.$$

- L^2 is a Hilbert space, i.e., any sequence of random variables $(X_n)_n$ converges in L^2 iff it is a Cauchy sequence.

Proof

We show $(M_n)_n$ is a Cauchy sequence in L^2. For any integers $n < p$,

$$\begin{aligned}\mathbb{E}\left[(M_n - M_p)^2\right] &= \mathbb{E}(M_n^2) + \mathbb{E}(M_p^2) - 2\underbrace{\mathbb{E}\left[M_p M_n\right]}_{=\mathbb{E}\left[\mathbb{E}[M_p M_n|\mathcal{F}_n]\right]} \\ &= \mathbb{E}(M_n^2) + \mathbb{E}(M_p^2) - 2\mathbb{E}\Big[M_n \underbrace{\mathbb{E}\left[M_p|\mathcal{F}_n\right]}_{=M_n}\Big] \\ &= \mathbb{E}(M_p^2) - \mathbb{E}(M_n^2).\end{aligned}$$

Let $\ell = \lim_{n\to\infty} \mathbb{E}(M_n^2)$. Fix $\varepsilon > 0$ and $N \in \mathbb{N}$ s.t. $\forall n \geq N$, $\left|\mathbb{E}(M_n^2) - \ell\right| \leq \varepsilon$.

Then $\forall n, p \geq N$, $\left|\mathbb{E}(M_n^2) - \mathbb{E}(M_p^2)\right| \leq \left|\mathbb{E}(M_n^2) - \ell\right| + \left|\ell - \mathbb{E}(M_p^2)\right| \leq 2\varepsilon$.

Thus, $(M_n)_n$ is a Cauchy sequence. Since L^2 is a Hilbert space, $(M_n)_n$ converges in L^2.

Uniform integrability

Reminder

Last time, we saw that a submartingale bounded in L^1 converges a.s.

Theorem (Reminder: Martingale Convergence Theorem)

Let $(X_n, \mathcal{F}_n)_{n \in \mathbb{Z}_+}$ be a submartingale and suppose that

$$\sup_{n \in \mathbb{Z}_+} \mathbb{E}\left[|X_n|\right] < \infty.$$

Then there exists an integrable r.v. X_∞ such that

$$\lim_{n \to \infty} X_n = X_\infty \quad \text{a.s.}$$

Now, we will see a stronger condition (uniform integrability) to obtain L^1 convergence.

Uniform integrability

Uniform integrability is a strengthening of $\sup_{n \in \mathbb{Z}_+} \mathbb{E}[|X_n|] < \infty$.

Definition

A sequence $X = (X_n)_n$ of random variables is uniformly integrable if

$$\lim_{c \to \infty} \sup_n \mathbb{E}\left[|X_n| \mathbf{1}_{|X_n| \geq c}\right] = 0.$$

Remarks (to check as exercises):

- A finite family of integrable random variables is uniformly integrable.
- If $(X_n)_n$ is bounded in L^2, then it is uniformly integrable.
- If $(X_n)_n$ is uniformly integrable, then $\sup_n \mathbb{E}|X_n| < \infty$.

Characterization of uniformly integrable random variables

The next proposition characterizes uniform integrability.

Proposition

The following assertions are equivalent

1. $(X_n)_n$ is uniformly integrable.
2. $\sup_n \mathbb{E}[|X_n|] < \infty$ and for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\mathbb{P}(A) \leq \eta \implies \sup_n \mathbb{E}\big[\,|X_n|\,\mathbf{1}_A\big] \leq \varepsilon. \tag{1}$$

In what follows, we will only need (1) ⇒ (2), but we include the proof of (2) ⇒ (1) for completeness.

Proof (1) ⇒ (2)

Suppose $X = (X_n)_n$ is uniformly integrable. There is $c > 0$ such that

$$\sup_n \mathbb{E}\left[|X_n| \mathbf{1}_{|X_n| \geq c}\right] \leq 1.$$

Then, we have

$$\mathbb{E}\left[|X_n|\right] \leq \mathbb{E}\left[|X_n| \mathbf{1}_{|X_n| \geq c}\right] + \mathbb{E}\left[|X_n| \mathbf{1}_{|X_n| \leq c}\right] \leq 1 + c.$$

Hence, $\sup_n \mathbb{E}\left[|X_n|\right] < \infty$.

Proof (1) ⇒ (2)

Moreover, for any event A,

$$\mathbb{E}\left[|X_n|\,\mathbf{1}_A\right] = \mathbb{E}\left[|X_n|\,\mathbf{1}_A\mathbf{1}_{|X_n|\geq c}\right] + \mathbb{E}\left[|X_n|\,\mathbf{1}_A\mathbf{1}_{|X_n|\leq c}\right]$$

For $\varepsilon > 0$, there exists c such that

$$\sup_n \mathbb{E}\left[|X_n|\,\mathbf{1}_A\mathbf{1}_{|X_n|\geq c}\right] \leq \frac{\varepsilon}{2}.$$

Therefore, we have $\mathbb{E}\left[|X_n|\,\mathbf{1}_A\right] \leq \frac{\varepsilon}{2} + c\,\mathbb{P}(A)$.

Choose $\eta = \frac{\varepsilon}{2c}$ to obtain $\mathbb{E}\left[|X_n|\,\mathbf{1}_A\right] \leq \varepsilon$.

Proof (2) $\Rightarrow$ (1)

Conversely, suppose $X = (X_n)_n$ is bounded in L^1 and satisfies (1).

By Markov's inequality, we have $\mathbb{P}(|X_n| \geq c) \leq \frac{M}{c}$.

Hence, given $\varepsilon > 0$ and $\eta > 0$ given by (1), we have for $c > \frac{M}{\eta}$,

$$\sup_n \mathbb{E}\left[|X_n| \mathbf{1}_{|X_n| \geq c}\right] \leq \varepsilon.$$

This concludes the proof.

Convergence in L^1

Convergence in L^1

We now state the main result regarding L^1 convergence.

Proposition

Let $X = (X_n)_n$ be uniformly integrable. Then, $(X_n)_n$ converges a.s. and in L^1 toward an integrable random variable X.

Elements for the proof

- (Fatou's lemma) If $\forall n \in \mathbb{N} : X_n \in [0, \infty]$, then

$$\mathbb{E}[\liminf_n X_n] \leq \liminf_n \mathbb{E}[X_n].$$

- Almost sure convergence implies convergence in probability.
- If $(X_n)_n$ is uniformly integrable, then

$$\forall \varepsilon > 0,\ \exists \eta > 0,\ \forall A \in \mathcal{B} : \left[\mathbb{P}(A) \leq \eta \Longrightarrow \sup_n \mathbb{E}\left[|X_n| \mathbf{1}_A\right] \leq \varepsilon\right].$$

Proof

$(X_n)_n$ is U.I. hence bounded in L^1, so $(X_n)_n$ converges a.s. to a random variable X. Let's first prove that X is integrable.

By Fatou's Lemma, since $\forall n \in \mathbb{Z}_+ : |X_n| \in [0, \infty]$ we have

$$\mathbb{E}\big[|X|\big] = \mathbb{E}\left[\lim_n |X_n|\right] \leq \liminf_n \mathbb{E}\left[|X_n|\right] \leq \sup_n \mathbb{E}\big[|X_n|\big] < \infty$$

hence X is integrable.

Let's now move to the L^1 convergence. Letting $\varepsilon > 0$, we have

$$\begin{aligned} \mathbb{E}\left[|X_n - X|\right] &\leq \mathbb{E}\left[|X_n - X| \, \mathbf{1}_{|X_n - X| \geq \varepsilon}\right] + \varepsilon. \\ &\leq \mathbb{E}\left[\big(|X_n| + |X|\big) \mathbf{1}_{|X_n - X| \geq \varepsilon}\right] + \varepsilon \end{aligned}$$

Proof

Since $X = (X_n)_n$ is uniformly integrable and X is integrable, $(|X_n| + |X|)_n$ is uniformly integrable, hence there exists $\eta > 0$ such that for any event A

$$\mathbb{P}(A) \leq \eta \Longrightarrow \mathbb{E}\left[\left(|X_n| + |X|\right)\mathbf{1}_A\right] \leq \varepsilon. \tag{1}$$

Since $X_n \xrightarrow{a.s.} X$, we also have $X_n \xrightarrow{\mathbb{P}} X$, hence we can fix $N \in \mathbb{N}$ such that for any $n \geq N$, we have $\mathbb{P}\left(|X_n - X| \geq \varepsilon\right) \leq \eta$.

Given (1), we then have for $n \geq N$,

$$\mathbb{E}\Big[\big|X_n - X\big|\Big] \leq \mathbb{E}\Big[\Big(\big|X_n\big| + \big|X\big|\Big)\mathbf{1}_{|X| \geq \varepsilon}\Big] + \varepsilon \leq 2\varepsilon.$$

This concludes the proof.

Concluding remarks

Assume further that the martingale $(M_n)_n$ is bounded in L^2.

We have seen that it is also uniformly integrable, so we have the following corollary.

Corollary

If $(M_n)_n$ is a martingale such that $\sup_n \mathbb{E}[M_n^2] < \infty$, then $(M_n)_n$ converges in L^1, L^2 and almost surely.

Application: Martingales and arbitrage

Problem statement

Goal: prove the simplest version of the fundamental theorem of mathematical finance which links a concept of finance (arbitrage) and a mathematical concept (martingales).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space with $\mathcal{F} = \mathcal{P}(\Omega)$.

We assume $\mathbb{P}(\omega) > 0$ for every $\omega \in \Omega$.

We endow Ω with a filtration $(\mathcal{F}_n)_{0 \leq n \leq N}$.

For simplicity, we assume that $\mathcal{F}_0$ is trivial: $\mathcal{F}_0 = \{\varnothing, \Omega\}$.

Problem statement

Definition (Reminders)

A sequence $(X_n)_n$ of real-valued R.V. is adapted (resp. predictable) if for every $n \in \mathbb{Z}_+$, X_n is $\mathcal{F}_n$-measurable (resp. $\mathcal{F}_{n-1}$-measurable).

Definition

Let $(M_n)_{0 \leq n \leq N}$ be a martingale and $(H_n)_n$ be predictable w.r.t. the filtration $(\mathcal{F}_n)_{0 \leq n \leq N}$.

The stochastic integral of $(M_n)_n$ w.r.t. $(H_n)_n$ is the process $(X_n)_n$ defined by

$$X_n = X_0 + \sum_{k=1}^{n} H_k \left(M_k - M_{k-1}\right)$$

where X_0 is $\mathcal{F}_0$ measurable, i.e., X_0 is constant.

Properties

Proposition

Suppose that, for any $n \in \mathbb{Z}_+$, H_n is bounded a.s. and let $(M_n)_n$ be an integrable and adapted process. Then

1. $(X_n)_n$ is a martingale w.r.t. $(\mathcal{F}_n)_n$.
2. $(M_n)_n$ is a martingale w.r.t. $(\mathcal{F}_n)_n$ iff for every predictable process $(H_n)_n$ such that each H_n is bounded a.s., we have
$$\mathbb{E}\left(\sum_{k=1}^{N} H_k \left(M_k - M_{k-1}\right)\right) = 0.$$
3. Let $(Z_n)_n$ be iid with $\mathbb{E}\left(Z_1\right) = 0$ and $\operatorname{var}\left(Z_1\right) = 1$. Setting $M_n = Z_1 + \ldots + Z_n$, we have
$$\mathbb{E}\left(\sum_{k=1}^{N} H_k \left(M_k - M_{k-1}\right)\right)^2 = \mathbb{E}\left[\sum_{k=1}^{N} H_k^2\right].$$

Proof

1. $(X_n)_n$ is clearly $\mathcal{F}_n$-adapted. Since H_k is bounded a.s. for any $k \in \mathbb{Z}_+$, we can let $C_k > 0$ such that $|H_k| \leq C_k$ for any $k \in \mathbb{Z}_+$. We have

$$\mathbb{E}|X_n| \leq \mathbb{E}|X_0| + \sum_{k=1}^{n} \mathbb{E}\Big[C_k\Big(|M_k| + |M_{k-1}|\Big)\Big] < \infty$$

since each M_k is integrable. Moreover, for any $n \in \mathbb{Z}_+$

$$\begin{aligned}
\mathbb{E}[X_{n+1}|\mathcal{F}_n] &= X_n + \mathbb{E}\left[H_{n+1}(M_{n+1} - M_n)|\mathcal{F}_n\right] \\
&= X_n + H_{n+1} \underbrace{E\left[(M_{n+1} - M_n)|\mathcal{F}_n\right]}_{=0} \\
&= X_n.
\end{aligned}$$

Hence, $(X_n)_n$ is a martingale.

Proof

2. If $(M_n)_n$ is an $\mathcal{F}_n$-martingale and $(H_n)_n$ is predictable and such that each H_n is bounded a.s., we have shown that $\left(\sum_{k=1}^n H_k(M_k - M_{k-1})\right)_n$ is a martingale, hence

$$\mathbb{E}\left(\sum_{k=1}^{N} H_k\left(M_k - M_{k-1}\right)\right) = 0. \tag{2}$$

Conversely, assume that (2) holds for any predictable and bounded $(H_n)_n$. For any $n \in \mathbb{Z}_+$, let $\Lambda \in \mathcal{F}_{n-1}$ and $(H_k)_k$ be such that $H_k = \mathbf{1}_{k=n}\mathbf{1}_\Lambda$. Then we have

$$\begin{aligned} 0 &= \mathbb{E}\left(\sum_{k=1}^{N} H_k\left(M_k - M_{k-1}\right)\right) = \mathbb{E}\left[\mathbf{1}_\Lambda\left(M_n - M_{n-1}\right)\right] \\ &= \int_\Lambda M_n - M_{n-1} d\mathbb{P}, \end{aligned}$$

hence $\mathbb{E}[M_n|\mathcal{F}_{n-1}] = M_{n-1}$ by definition of the conditional expectation.

Proof

3. We have

$$\begin{aligned}
&\mathbb{E}\left(\sum_{k=1}^{N} H_k\left(M_k - M_{k-1}\right)\right)^2 \\
&= \mathbb{E}\left(\sum_{k=1}^{N} H_k Z_k\right)^2 = \sum_{j,k=1}^{N} \mathbb{E}\left[H_k H_j Z_k Z_j\right] \\
&= \sum_{k=1}^{N} \mathbb{E}\Big[H_k^2 \underbrace{\mathbb{E}[Z_k^2|\mathcal{F}_{k-1}]}_{=1}\Big] + 2\sum_{j<k=1}^{N} \mathbb{E}\Big[H_k H_j Z_j \underbrace{\mathbb{E}[Z_k|\mathcal{F}_{k-1}]}_{=0}\Big] \\
&= \mathbb{E}\left[\sum_{k=1}^{N} H_k^2\right].
\end{aligned}$$

This concludes the proof of the proposition.

Financial model

We now describe our simple financial market model.

We consider $d+1$ financial assets with prices at time n given by $S_n = \left(S_n^0, S_n^1, \ldots, S_n^d\right)$.

The asset labeled 0 is the risk-free asset. Letting r denote the one-period risk-free interest rate, we have $S_n^0 = (1+r)^n$.

We define the discounted value of the asset as $\tilde{S}_n^i = \frac{S_n^i}{S_n^0}$.

Portfolio

A portfolio is defined by a predictable process

$$(\phi_n)_n = ((\phi_n^0, \phi_n^1, \ldots, \phi_n^d))_{0 \le n \le N}.$$

Here, ϕ_n^i is the number of shares of asset i held in the portfolio.

The portfolio value at time n is defined as $V_n = \langle \phi_n, S_n \rangle$ and the discounted portfolio value as $\tilde{V}_n = \langle \phi_n, \tilde{S}_n \rangle$.

We now introduce a very important notion.

Definition

A portfolio strategy is self-financed if for every $n \in \{0, 1, \ldots, N-1\}$, we have

$$\langle \phi_n, S_n \rangle = \langle \phi_{n+1}, S_n \rangle.$$

Some useful properties

Proposition

The following assertions are equivalent. Let $V_0 \in \mathbb{R}$ be given.

1. A portfolio strategy is self-financed.
2. $\forall n \in \{1, \dots, N\}$: $V_n(V_0, \phi) = V_0 + \sum_{k=1}^{n} \langle \phi_k, S_k - S_{k-1} \rangle$.
3. $\forall n \in \{1, \dots, N\}$: $\tilde{V}_n(V_0, \phi) = V_0 + \sum_{k=1}^{n} \langle \phi_k, \tilde{S}_k - \tilde{S}_{k-1} \rangle$.

Proof

(1) $\Rightarrow$ (2). Suppose the portfolio strategy is self-financed. Then, for any $n \in \{1, \ldots, N\}$, we have

$$\begin{aligned} V_n = \langle \phi_n, S_n \rangle &= \underbrace{\langle \phi_0, S_0 \rangle}_{=V_0} + \sum_{k=1}^{n} \langle \phi_k, S_k \rangle - \underbrace{\langle \phi_{k-1}, S_{k-1} \rangle}_{=\langle \phi_k, S_{k-1} \rangle} \\ &= V_0 + \sum_{k=1}^{n} \langle \phi_k, S_k - S_{k-1} \rangle. \end{aligned}$$

The proof of (1) $\Rightarrow$ (3) is similar.

Proof

(2) $\Rightarrow$ (1). From (2), we have for any $n \in \mathbb{Z}_+$:

$V_{n+1} - V_n = \langle \phi_{n+1}, S_{n+1} - S_n \rangle, \quad$ i.e. $\quad V_{n+1} = V_n + \langle \phi_{n+1}, S_{n+1} - S_n \rangle$

This implies

$$\langle \phi_{n+1}, S_{n+1} \rangle = \langle \phi_n, S_n \rangle + \langle \phi_{n+1}, S_{n+1} - S_n \rangle$$

i.e.

$$\langle \phi_{n+1}, S_n \rangle = \langle \phi_n, S_n \rangle,$$

by rearranging the terms. The proof of (3) $\Rightarrow$ (1) is similar.

Arbitrage

Moving forward to the notion of arbitrage, we introduce a second notion regarding portfolios.

Definition

A portfolio strategy is admissible if it is self-financed and satisfies $V_n \geq 0$ for $n \in \{0, 1, \ldots, N\}$.

The notion of arbitrage (making a profit without running a risk) is then defined as follows

Definition

A portfolio strategy is an arbitrage if it is admissible with $V_0 = 0$ and $\mathbb{P}(V_N > 0) > 0$.

Fundamental theorem of mathematical finance

Theorem

A financial market is arbitrage-free if and only if there exists a probability measure $\mathbb{P}^*$ under which the discounted asset prices are martingales.

We split the proof into two parts: Necessary and sufficient condition.

Sufficient condition

Assume that the discounted prices $\tilde{S}_n^i$ are martingales under a probability $\mathbb{P}^*$.

Proposition

Any discounted portfolio is a martingale under $\mathbb{P}^*$. It follows that $V_0 = 0$ implies $V_N = 0$ and, therefore, the market is arbitrage-free.

Proof

Suppose $(\tilde{V}_n)_n$ is a martingale under $\mathbb{P}^*$.

Since by definition, we have $V_n \geq 0$ a.s., we have both $\mathbb{E}\tilde{V}_N = \mathbb{E}V_0 = 0$ and $\tilde{V}_N \geq 0$ a.s.

This imposes that $V_N = 0$ $\mathbb{P}^*$ a.s.

Necessary condition

Let Γ be the set of nonzero positive random variables. Γ is a convex cone, and a market is arbitrage-free if

$$V_0 = 0 \to \tilde{V}_N(0, \phi) \notin \Gamma$$

For any predictable strategy $\phi_n = \left(\phi_n^1, \ldots, \phi_n^d\right)$, we define the process of discounted profit and losses as follows

$$\tilde{G}_n(\phi) = \sum_{k=1}^{n} \langle \phi_k, \tilde{S}_k - \tilde{S}_{k-1} \rangle$$

Observe that for any admissible portfolio strategy ϕ, we have $G_n(\phi) = \tilde{V}_n(V_0, \phi)$. Therefore, if the market is arbitrage-free then $\tilde{G}_N(\phi) \notin \Gamma$ for every admissible portfolio ϕ.

Necessary condition

We now define the subspace $\mathcal{G}$ of $\mathbb{R}^\Omega$ (set of real-valued random variables defined on Ω), which consists of the random variables of type $\tilde{G}_N(\phi)$ with ϕ predictable with values in $\mathbb{R}^d$.

If the market is arbitrage-free, we deduce from above that $\mathcal{G}$ does not intersect the convex compact set $K = \{X \in \Gamma \mid \sum_\omega X(\omega) = 1\}$.

Lemma

Let $d \in \mathbb{N}$, V be a linear subspace of $\mathbb{R}^d$ and $K \subset \mathbb{R}^d$ be compact and convex. If $K \cap V = \varnothing$, then there exists a vector $\lambda \in \mathbb{R}^d$ such that

$$\forall v \in V,\ \langle \lambda, v \rangle = 0 \quad \text{and} \quad \forall x \in K,\ \langle \lambda, x \rangle > 0.$$

Proof

For any $x \in \mathbb{R}^d$, let $d(x, V) := \arg\min_{v \in V} \|x - v\|$ (distance from x to V).

Recall that $d(\cdot, V)$ is continuous. Since K is compact, $d(\cdot, V)$ attains its minimum over K, so we can fix $x_0 \in \arg\min_{x \in K} d(x, V) \neq \emptyset$.

Then we have $d(x_0, V) = \|\Pi_{V^\perp}(x_0)\|$ where $\Pi_{V^\perp}$ is the orthogonal projector onto $V^\perp$.

Proof

Let $x \in K$ and define $\phi(t) = tx + (1-t)x_0, \forall t \in [0,1]$.

Since K is cvx and $x, x_0 \in K$, we have $\phi(t) \in K, \forall t \in [0,1]$. Consider

$$\psi(t) = \left\|\Pi_{V^\perp}\big(\phi(t)\big)\right\|^2 = d\big(\phi(t), V\big)^2.$$

This function is differentiable and we have

$$\psi'(t) = 2\Big\langle \Pi_{V^\perp}\big(\phi(t)\big),\ \phi'(t)\Big\rangle, \quad \text{hence} \quad \psi'(0) = 2\Big\langle \Pi_{V^\perp}\big(x_0\big),\ x - x_0\Big\rangle.$$

Proof

By definition of $x_0 = \arg\min_{y \in K} d(y, V)$, we have

$$\psi(t) = d(\phi(t), V)^2 \geq d(x_0, V)^2 = \psi(0), \quad \forall t \in [0, 1].$$

It follows that $\psi'(0) = \lim_{h \to 0} \frac{\psi(h) - \psi(0)}{h} \geq 0$, that is

$$\begin{aligned} & \left\langle \Pi_{V^\perp}(x_0), x - x_0 \right\rangle \geq 0 \\ \iff \quad & \left\langle \Pi_{V^\perp}(x_0), x \right\rangle \geq \left\langle \Pi_{V^\perp}(x_0), x_0 \right\rangle \\ & \qquad = \left\langle \Pi_{V^\perp}(x_0), \Pi_{V^\perp}(x_0) \right\rangle + \cancel{\left\langle \Pi_{V^\perp}(x_0), \Pi_V(x_0) \right\rangle} \\ & \qquad = \left\| \Pi_{V^\perp}(x_0) \right\|^2. \end{aligned}$$

Proof

To conclude the proof, it suffices to let $\lambda = \Pi_{V^\perp}(x_0)$.

Indeed, we have $\lambda \in V^\perp$, i.e. $\forall v \in V : \langle \lambda, V \rangle = 0$, as desired.

Moreover, we have $\lambda \neq 0$ since $\|\lambda\| = d(x_0, V)$ where $x_0 \in K$ and $K \cap V = \varnothing$.

Lastly, we have proved that for any $x \in K$,

$$\langle \lambda, x \rangle = \left\langle \Pi_{V^\perp}(x_0), x \right\rangle \geq \left\| \Pi_{V^\perp}(x_0) \right\|^2 = \|\lambda\|^2 > 0$$

since $\lambda \neq 0$ as argued above.

This concludes the proof.

Necessary condition

We now apply the previous lemma to the linear set $V = \mathcal{G}$ and the compact set $K = \{X \in \Gamma \mid \sum_\omega X(\omega) = 1\}$.

Corollary

There exists $(\lambda(\omega))_\omega$ such that

1. $\forall X \in K : \sum_\omega \lambda(\omega) X(\omega) > 0$
2. for all predictable ϕ,

$$\sum_\omega \lambda(\omega) \tilde{G}_N(\phi)(\omega) = 0$$

Therefore, we can construct a probability measure $\mathbb{P}^*$ defined as

$$\mathbb{P}^*(\omega_0) = \frac{\lambda(\omega_0)}{\sum_\omega \lambda(\omega)}$$

under which the discounted asset prices are martingales.

Proof

The existence of λ and thus of $\mathbb{P}^*$ follows from the previous proposition.

Now, we have found a probability P^* independent of ϕ such that, for any predictable process $(\phi_n)_n$, we have

$$\mathbb{E}_{\mathbb{P}^*}\left[\sum_{k=1}^{n}\langle \phi_k, \tilde{S}_k - \tilde{S}_{k-1}\rangle\right] = 0.$$

We have seen that this implies that $(\tilde{S}_n)_n$ is an $(\mathcal{F}_n, \mathbb{P}^*)$-martingale.

An application

Let's consider a special case. Let $(Z_n)_{1\leq n\leq N}$ be i.i.d. and such that

$$p = \mathbb{P}\left(Z_1 = h\right) = 1 - \mathbb{P}\left(Z_1 = l\right) \text{ with } h > 1 > \ell > 0.$$

Assume the market has only one risky asset $(S_n)_{0\leq n\leq N}$ and one riskless asset $S_n^0 = (1+r)^n$.

Let $S_0 > 0$ and, for any $n \geq 1$, define $S_n = S_0 Z_1 \ldots Z_n$.

Proposition

1. There exists a unique probability measure such that $\left(\tilde{S}_n\right)_{0\leq n\leq N}$ is an $\mathcal{F}_n$-martingale.
2. There exist a value V_0 and a strategy ϕ such that

$$V_N\left(V_0, \phi\right) = \left(S_N - 1\right)_+.$$

The value V_0 is the value of a call option with strike price 1.

Proof

$(\tilde{S}_n)_n$ is a martingale if, and only if,

$$\forall n \in \{0, N-1\} : \quad \mathbb{E}\left[\tilde{S}_{n+1} | \mathcal{F}_n\right] = \tilde{S}_n$$

$$\iff \quad \frac{\tilde{S}_n}{(1+r)^{n+1}} \mathbb{E}\left[Z_{n+1} | \mathcal{F}_n\right] = \frac{\tilde{S}_n}{(1+r)^n}$$

$$\iff \quad \mathbb{E}\left[Z_{n+1} | \mathcal{F}_n\right] = 1 + r$$

$$\iff \quad ph + (1-p)\ell = 1 + r$$

$$\iff \quad p = \frac{1 + r - \ell}{h - \ell}.$$

Hence, there exists a unique probability measure such that $(\tilde{S}_n)_n$ is an $\mathcal{F}_n$-martingale.

Proof

Let $n \in \{0, \dots, N-1\}$. For any $n \in \mathbb{Z}_+$, we must have

$$V_{n+1} - V_n = \langle \phi_{n+1}, S_{n+1} - S_n \rangle.$$

We define

$$V_{n+1}^{\uparrow} = \langle \phi_{n+1}, S_n h - S_n \rangle \quad \text{and} \quad V_{n+1}^{\downarrow} = \langle \phi_{n+1}, S_n \ell - S_n \rangle.$$

We let $\phi_n = (a_n, b_n)$, so that the portfolio value at any time is $V_n = a_n S_n^0 + b_n S_n$. Then we have

$$V_{n+1}^{\uparrow} - V_{n+1}^{\downarrow} = b_{n+1} S_n (h - \ell) \quad \text{hence} \quad b_{n+1} = \frac{V_{n+1}^{\uparrow} - V_{n+1}^{\downarrow}}{S_n (h - \ell)}.$$

Moreover,

$$a_{n+1} = \frac{V_{n+1}^{\uparrow} - b_{n+1} S_n h}{S_{n+1}^0} = \frac{V_{n+1}^{\downarrow}}{S_{n+1}^0}$$

Proof

We have constructed a predictable process $\phi = (\phi_n)_n = ((a_n, b_n)_n)$ such that

$$\forall n \in \mathbb{Z}_+ : V_n = V_0 + \sum_{k=1}^{n} \langle \phi_k, S_k - S_{k-1} \rangle$$

hence the strategy is self-financed, as proved above.

It suffices to initialize ϕ_N such that $b_N = \frac{V_N^{\uparrow} - V_N^{\downarrow}}{S_{N-1}(h-\ell)}$ and $a_N = \frac{V_N^{\downarrow}}{S_N^0}$ and define (a_n, b_n) backward recursively.

Since $(\tilde{S}_n)_n$ is an $(\mathcal{F}_n, \mathbb{P}^*)$-martingale, we may set

$$V_0 = \mathbb{E}_{\mathbb{P}^*}\left[\tilde{V}_N\right] = \mathbb{E}_{\mathbb{P}^*}\left[\frac{(S_N - 1)_+}{(1+r)^N}\right].$$