Exercise Session: Sampling

Exercise 1: Classical Sampling Methods

1. Apply the inversion of the cdf method to simulate a random variable Y with a Weibull distribution of density $3x^2e^{-x^3}$ over $\mathbb{R}^+$.
2. Compute the distribution of $C\sqrt{-\ln(U)}$, if $U \sim \text{Unif}([0,1])$ and for any constant $C > 0$. Deduce a way to simulate a Rayleigh distribution, with density $\frac{x}{\sigma^2}e^{-x^2/2\sigma^2}$ over $\mathbb{R}^+$.

Exercise 2: Other Integrals

Write the following integrals as an expectation, and provide Monte-Carlo methods to estimate them:

1. $\int_0^1 \sin(\sqrt{x})dx$.
2. $\int_0^{+\infty} \sin(x^4)\exp(-2x)\exp(-x^2/2)dx$ (propose two methods for this integral).

Exercise 3: Importance sampling

1. Suppose that p is the $\mathcal{N}(0,1)$ distribution, and that $f(x) = \exp\left(-\frac{(x-10)^2}{2}\right)$. Find the optimal importance sampling density q.

   Hint: a random variable X is distributed as $N(\mu, \sigma^2)$ iff its density is *proportional to* $x \mapsto \exp\big(-\frac{(x-\mu)^2}{2\sigma^2}\big)$.
2. Suppose that p is the $\mathcal{N}(0,1)$ distribution, and that f is $\exp(kx)$ for $k \neq 0$. Find the optimal importance sampling density q.
3. Let p be the $\mathcal{N}(0, I)$ distribution in dimension $d \geq 1$.
   (a) Generalize Question 1 to the case where f is the density of $\mathcal{N}(\theta, I)$.
   Hint: a random variable $X \in \mathbb{R}^d$ is distributed as $N(\mu, \sigma^2 I)$ iff its density is *proportional to* $x \mapsto \exp\big(-\frac{\|x-\mu\|^2}{2\sigma^2}\big)$.
   (b) Generalize Question 2 to the case where $f(x) = \exp(k^\top x)$ for $k \in \mathbb{R}^d$.

Exercise 4: Rejection method

We define the following quantities.

- Let X be random variable with probability density $f(x) = \frac{1}{Z}\exp(-x^3)$ over $\{x \in \mathbb{R} : x \geq 1\}$. Here, the normalizing constant $Z = \int_1^{+\infty} e^{-x^3}dx$ is not given explicitly, but its value does not matter for this exercise.
- Let Y be a random variable with probability density $g(x) = \frac{1}{x^2}$ over $\{x \in \mathbb{R} : x \geq 1\}$.

1. Give a procedure to simulate Y using the inversion of the CDF method.

2. Show that the function $x \mapsto \frac{f(x)}{g(x)}$ is decreasing over $\{x \in \mathbb{R} : x \geq 1\}$.
3. Deduce that $f(x) \leq \dfrac{1}{eZ} g(x)$ for any $x \geq 1$.
4. Deduce a rejection algorithm to simulate $X \sim f$ using the auxiliary law g.