Exercise Session: MCMC methods

Exercise 1 (Simulation of a Pair)

Let (X, Y) be a random pair with joint density

$$f(x, y) = e^{-y} \mathbf{1}_{0 \leq x \leq y}.$$

1. Determine the marginal density of X, denoted $f(x)$. What distribution do you recognize?

2. Given $X = x \geq 0$, determine the conditional density $f(y|x)$. What distribution do you recognize?

3. Deduce a method to simulate a realization of the random pair (X, Y).

4. Given $Y = y \geq 0$, determine the conditional density $f(x|y)$. What distribution do you recognize?

5. Starting, for example, from the point $(x_0, y_0) = (0, 1)$, write the pseudo-code of a Gibbs sampler with deterministic scanning to obtain a sample $(X_1, Y_1) \ldots, (X_n, Y_n)$ to sample from the target law f.

6. Between the two proposed methods, which one would you choose to simulate according to the density $f(x, y)$?

Exercise 2 (Metropolis-Hastings Algorithm)

Let E be a finite state space and π a probability measure on E such that $\pi(x) > 0$ for any $x \in E$. We recall the Metropolis-Hastings algorithm for an auxiliary transition kernel Q (such that $Q(x, y) = 0$ if and only if $Q(y, x) = 0$)

Step 0:
 `Initialize` X_0 ;
Step n+1:
 `Choose` y `according to the law` $Q(X_n, \cdot)$;
 `Set` $X_{n+1} = y$ `with probability` $\min\left(1, \frac{\pi(y)Q(y, X_n)}{\pi(X_n)Q(X_n, y)}\right)$, `otherwise set` $X_{n+1} = X_n$

1) Show that if the acceptance-rejection probability $\min\left(1, \frac{\pi(y)Q(y,x)}{\pi(x)Q(x,y)}\right)$ is replaced by

$$\frac{\pi(y)Q(y, x)}{\pi(y)Q(y, x) + \pi(x)Q(x, y)},$$

the measure π will still be invariant for the Markov chain defined by the Metropolis-Hastings algorithm.

2) More generally, replacing the acceptance-rejection probability $\min\left(1, \frac{\pi(y)Q(y,x)}{\pi(x)Q(x,y)}\right)$ by

$$\alpha\left(\frac{\pi(y)Q(y, x)}{\pi(x)Q(x, y)}\right),$$

with $\alpha : \mathbb{R}_+ \to (0, 1]$, give a sufficient condition on the function α for the measure π to be the invariant measure of the Markov chain defined by the Metropolis-Hastings algorithm.

Exercise 3 (Joint law and Gibbs sampler)

We consider the density

$$f(x,y) = C \exp\left(-\frac{y^2}{2} - \frac{x^2(y^2 + 4y + 4)}{2}\right).$$

The renormalizing constant $C > 0$ is not specified here, but it does not matter for this exercise.

1. What is the law of X conditional on $Y = y$? What is the law of Y conditional on $X = x$?
2. Propose a deterministic scan Gibbs sampler pseudo-code to sample from the target f.

Exercise 4 (Markov chain)

For $\theta \in [0, \frac{1}{2}]$, we consider the Markov chain on $E = \{1, 2, 3\}$ with initial state $X_0 = 1$ and transition matrix

$$\Pi = \begin{pmatrix} \frac{1}{2} & \frac{\theta}{2} & \frac{1}{2}(1-\theta) \\ \theta & 1-2\theta & \theta \\ 0 & 1 & 0 \end{pmatrix}.$$

1. For what values of θ is the chain irreducible?

2. Suppose $\theta = 0$ and $X_1 = 1$. Compute $\mathbb{P}(X_3 = 1 | X_1 = 1)$.

3. Let $k \in \{1, 2, 3\}$. Suppose $\theta = \frac{1}{3}$. What is the almost sure limit of $\frac{1}{n} \sum_{j=1}^{n} \mathbf{1}_{X_j = k}$?

Exercise 5 (Simultaneous Gibbs = Flawed Gibbs)

Consider the distribution $\pi = \pi_{X,Y}$ over $E = \{(0,0), (0,1), (1,0), (1,1)\}$ defined as follows

$$\pi(0,0) = \frac{2}{5}, \quad \text{and} \quad \pi(0,1) = \pi(1,0) = \pi(1,1) = \frac{1}{5}.$$

On this state space, we consider the Markov chain (X_n) defined by the following transitions: Letting $(x,y) = (X_t, Y_t)$, draw independently

$$X_{t+1} \sim \pi_{X|Y}(\cdot \mid y) \quad \text{and} \quad Y_{t+1} \sim \pi_{Y|X}(\cdot \mid x)$$

This way of moving in the state space is thus akin to a Gibbs sampler where all coordinates are updated simultaneously (that is to say, not using the most recent updates available so far).

1. Compute the probability distributions $\pi_{X|Y}(\cdot \mid y)$ and $\pi_{Y|X}(\cdot \mid x)$ for any $x, y \in \{0, 1\}$ and show that $\pi_{X|Y} = \pi_{Y|X}$. Therefore, we will write

$$\pi_{a,b} := \pi_{X|Y}(a \mid b) = \pi_{Y|X}(a \mid b), \quad \forall a, b \in \{0, 1\}.$$

2. Justify that, for any $x, x', y, y' \in \{0, 1\}$, we have

$$\mathbb{P}\left((X_{t+1}, Y_{t+1}) = (x', y') \mid (X_t, Y_t) = (x, y)\right) = \pi_{x'y} \, \pi_{y'x}.$$

3. Show that the transition matrix P of the Markov chain (X_n) is

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$
$(0,1)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$
$(1,0)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
$(1,1)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Is this chain irreducible? Is it aperiodic?

4. Show that the proposed algorithm does not converge to the desired probability density.

5. Show that the transition matrix P_a for the Gibbs sampler with random scanning is

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	0
$(0,1)$	$\frac{1}{3}$	$\frac{5}{12}$	0	$\frac{1}{4}$
$(1,0)$	$\frac{1}{3}$	0	$\frac{5}{12}$	$\frac{1}{4}$
$(1,1)$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

Verify that π is stationary for P_a. Is the chain reversible?